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Approximating Measures Invariant under Higher-Dimensional Chaotic Transformations

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Let τ be a Jablonski transformation from the n-dimensional unit cube into itself. We present a method for approximating the absolutely continuous invariant measures by means of approximating the Frobenius-Perron operator by finite-dimensional operators. This proves an n-dimensional version of a conjecture by Ulam and generalizes the one-dimensional results of T. Y. Li.

1. INTRODUCTION

Let $I = [0, 1]$ and let $\tau: I^n \to I^n$ be a piecewise expanding transformation. For $n = 1$, Lasota and Yorke [12] proved the existence of an absolutely continuous invariant measure (acim) μ with respect to Lebesgue measure. Since then there have been a number of generalizations of this result to higher dimensions [3, 5, 10, 16]. If f is the density of μ with respect to Lebesgue measure m on I^n, then it is well known that f is the fixed point of the Frobenius–Perron operator P_τ. However, solving the resulting functional equation $P_\tau f = f$ is infeasible in all except the most trivial cases.

In [17, p. 75], Ulam conjectured that it was possible to construct finite-dimensional operators which approximate P_τ and whose fixed points approximate the fixed point of P_τ. In [13] this conjecture was proved for a class of one-dimensional piecewise expanding transformations.

The aim of this paper is to prove a version of Ulam's conjecture in a higher-dimensional setting. The main difficulty in extending the method of [13] is due to the definition of bounded variation in n dimensions which is complicated and does not possess the same intuitive properties as one-dimensional bounded variation [7]. We shall restrict our attention to a special class of higher-dimensional transformations which we shall refer to

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as Jablonski transformations. Such transformations are defined on rectangular partitions of I^n and on each element of such a partition each component of τ depends only on one variable. In spite of these restrictions, the Jablonski transformations are nontrivial extensions of the one-dimensional transformations. The Jablonski transformations are L^1 dense in the class of all piecewise expanding transformations on I^n [15]. Recently, these transformations have found an interesting application to cellular automata [6], where they are used to model the dynamics on the space of configurations.

In Section 2 we introduce the Tonelli definition of bounded variation for higher-dimensional functions [9] and state the existence theorem of Jablonski [9]. In Section 3, we obtain a generalization of the main result of [13] to Jablonski transformations in n dimensions. Unlike the strong convergence in one dimension, our result provides a weak approximation to the invariant functions. In Section 4, we discuss uniqueness of absolutely continuous invariant measures for Jablonski transformations and in Section 5 we present examples.

2. Jablonski Transformations

Let $I^n = [0, 1]^n$ and let m_j denote Lebesgue measure on I^j. For $j = n$, let $m = m_n$. We let L^1 denote the space of all Lebesgue integrable functions on I^n. The transformation $\tau\colon I^n \to I^n$ is written as

$$\tau(x_1, ..., x_n) = (\varphi_1(x_1, ..., x_n), ..., \varphi_n(x_1, ..., x_n)),$$

where for any $i = 1, ..., n$, $\varphi_i(x_1, ..., x_n)$ is a function from I^n into $[0, 1]$.

We say that a measurable transformation $\tau\colon I^n \to I^n$ is nonsingular if $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. For nonsingular $\tau\colon I^n \to I^n$, we define the Frobenius–Perron operator $P_\tau : L^1 \to L^1$ by the formula

$$\int_A P_\tau f\, dx = \int_{\tau^{-1}(A)} f\, dx,$$

where $A \subseteq I^n$ is measurable. It follows that for $x = (x_1, ..., x_n)$,

$$P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(\prod_{i=1}^n [0, x_i])} f(y)\, dy.$$

It is well known that the operator P_τ is linear and satisfies the following conditions: P_τ is positive; P_τ preserves integrals; $P_{\tau^k} = P_{\tau^k}$, where τ^k denotes the nth iterate of τ and $P_\tau f = f$ if and only if the measure $d\mu = f\, dm$ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable subset A of I^n.

Let $\beta = \{D_1, ..., D_p\}$ be a partition of I^n such that $p < \infty$, i.e.,

$$\bigcup_{j=1}^{p} D_j = I^n, \qquad D_j \cap D_k = \varnothing \text{ for } j \neq k.$$

A partition β of I^n is called rectangular if for any $i \leqslant j \leqslant p$, D_j is an n-dimensional rectangle.

Definition 1. A transformation $\tau\colon I^n \to I^n$ is called a *Jablonski transformation* if it is defined on a rectangular partition and is given by the formula

$$\tau(x_1, ..., x_n) = (\varphi_{1j}(x_1), ..., \varphi_{nj}(x_n)),$$

where $(x_i, ..., x_n) \in D_j$, $1 \leqslant j \leqslant p$, $D_j = \prod_{i=1}^n [a_{ij}, b_{ij})$, and $\varphi_{ij} = [a_{ij}, b_{ij}] \to [0, 1]$. If $b_{ij} = 1$ for some i, then $[a_{ij}, b_{ij})$ means $[a_{ij}, b_{ij}]$.

Denote by $\prod_{i=1}^n A_i$ the Cartesian product of the sets A_i and by P_i the projection of R^n onto R^{n-1} given by

$$P_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Let $g\colon A \to R$ be a function on the n-dimensional interval $A = \prod_{i=1}^n [a_i, b_i]$. Fixing i, we define a function $\bigvee_i^A g$ of the $n-1$ variables $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ by the formula

$$\bigvee_i^A g = \bigvee_i g = \sup\left\{\sum_{k=1}^{r} |g(x_1, ..., x_i^k, ..., x_n) - g(x_1, ..., x_i^{k-1}, ..., x_n)| :\right.$$

$$\left. a_i = x_i^0 < x_i^1 < \cdots < x_i^r = b_i,\ r \in N\right\}.$$

For $f\colon A \to R$, where $A = \prod_{i=1}^n [a_i, b_i]$, let

$$\mathbb{V}_i^A f = \inf\left\{\int_{P_i(A)} \bigvee_i g \, dm_{n-1} : g = f \text{ almost everywhere, } \bigvee_i g \text{ measurable}\right\}$$

and let $\mathbb{V}^A f = \sup_{1 \leqslant i \leqslant n} \mathbb{V}_i^A f$. If $\mathbb{V}^A f < \infty$, then we say f is a bounded variation function on A and its total variation is $\mathbb{V}^A f$.

Theorem 1 [9]. *Let the Jablonski transformation $\tau\colon I^n \to I^n$ on the partition $\{D_j\}_{j=1}^p$ be given by*

$$\tau(x_1, ..., x_n) = (\varphi_{1j}(x_1), ..., \varphi_{nj}(x_n)),\ (x_1, ..., x_n) \in D_j,$$

where $D_j = [a_{ij}, b_{ij})$ *if* $b_{ij} < 1$ *and* $D_{ij} = [a_{ij}, b_{ij}]$ *if* $b_{ij} = 1$, $\varphi_{ij}\colon [a_{ij}, b_{ij}] \to [0, 1]$ *are* C^2 *functions, and*

$$\inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 1.$$

Then for any $f \in L^1$ *the sequence* $\{(1/l) \sum_{k=0}^{l-1} P_\tau^k f\}$ *is convergent in norm to a function* $f^* \in L^1$ *as* $l \to \infty$. *The limit function has the following properties:*

(1) $f \geqslant 0$ *implies* $f^* \geqslant 0$; (2) $\int_{I^n} f^* \, dm = \int_{I^n} f \, dm$;

(3) $P_\tau f^* = f^*$ *and consequently the measure* $d\mu^* = f^* \, dm$ *is invariant under* τ (f^* *is called an invariant density*); (4) *the function* f^* *is of bounded variation. Moreover, there exists a constant* C *independent of the choice of initial* f *such that the variation of the limiting* f^* *satisfies the inequality*

$$\bigvee f^* \leqslant C \, \| f \|.$$

3. Approximating the Invariant Densities

Let $\tau\colon I^n \to I^n$ be a Jablonski transformation and for any positive integer l, let I^n be divided into l^n subsets of equal measure $I_1, I_2, \ldots, I_{l^n}$ with

$$I_k = \left[\frac{r_1}{l}, \frac{r_1 + 1}{l}\right) \times \left[\frac{r_2}{l}, \frac{r_2 + 1}{l}\right) \times \cdots \times \left[\frac{r_n}{l}, \frac{r_n + 1}{l}\right)$$

for some $r_1, r_2, \ldots, r_n = 0, 1, \ldots, l-1$ and $m(I_k) = 1/l^n$, $k = 1, 2, \ldots, l^n$.

Let P_{st} be the fraction of I_s which is mapped into I_t by τ, i.e.,

$$P_{st} = m(I_s \cap \tau^{-1}(I_t))/m(I_s).$$

Let Δ_l be the l^n-dimensional linear subspace of L^1 which is the finite-dimensional space generated by $\{\chi_k\}_{k=1}^{l^n}$, where χ_k denotes the characteristic function of I_k, i.e., $f \in \Delta_l$ if and only if $f = \sum_{k=1}^{l^n} a_k \chi_k$ for some constants $a_1, a_2, \ldots, a_{l^n}$.

Define a linear operator $P_l = P_l(\tau)\colon \Delta_l \to \Delta_l$ by

$$P_l(\tau)\, \chi_k = \sum_{t=1}^{l^n} P_{kt} \chi_t.$$

Lemmas 1–5 are straightforward n-dimensional extensions of results in [13].

Lemma 1. *Let* $\Delta_l^1 = \{\sum_{k=1}^{l^n}) a_k \chi_k : a_k \geqslant 0$ *and* $\sum_{k=1}^{l^n} a_k = 1\}$. *Then* P_l *maps* Δ_l^1 *to a subset of* Δ_l^1.

DEFINITION 2. For $f \in L^1$ and for any positive integer l we define $Q_l: L^1 \to \Delta_l$ by $Q_l f = \sum_{k=1}^{l^n} C_k \chi_k$, where $m(I_k) = 1/l^n$ and

$$C_k = \frac{1}{m(I_k)} \int_{I_k} f(x)\, dx = l^n \int_{I_k} f(x)\, dx.$$

LEMMA 2. *If $f \in L^1$ then the sequence $Q_l f$ converges in L^1 to f as $l \to \infty$.*

LEMMA 3. *If $f \in \Delta_l$, then $P_l f = Q_l P_\tau f$.*

LEMMA 4. *If $f \in \Delta_l$ then the sequence $\{P_l f\}$ converges to $P_\tau f$ in L^1 as $l \to \infty$.*

LEMMA 5. *For any integer l there exists $f_l \in \Delta_l$ such that $P_l f_l = f_l$ and $\|f_l\| = 1$; i.e., P_l has a fixed point of norm 1.*

In the course of proving Thorem 1, the following result is established in [9].

THEOREM 2. *Let τ be a Jablonski transformation, where*

$$\tau(x) = (\varphi_{1j}(x_1), ..., \varphi_{nj}(x_n)),\ x \in D_j.$$

If $\lambda = \inf_{i,j} \{\inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}|\} > 2$, then for any $f \in L^1$,

$$\bigvee^{I^n} P_\tau f \leqslant K_\tau \|f\| + \alpha \bigvee^{I^n} f,$$

where K_τ is a constant depending on τ and $\alpha = 2\lambda^{-1} < 1$.

We require two more lemmas before we can prove our approximation result.

LEMMA 6. *If $f \in L^1$, then $\bigvee^{I^n} Q_l f \leqslant \bigvee^{I^n} f$.*

Proof. Let $I_k = \prod_{i=1}^n [(r_i/l), (r_i + 1/l)) = \prod_i^n {}_1 J_{r_i}$ for some $r_i = 0, 1, ..., l-1$, $k = 1, 2, ..., l^n$ and $m(I_k) = \prod_{i=1}^n m(J_{r_i})$. Let

$$Q_{l_i} f(x) = \sum_{r_i=0}^{l-1} \left(\frac{1}{m(J_{r_i})} \int_{J_{r_i}} f(x)\, dx_i \right) \chi_{J_{r_i}}(x_i).$$

Then

$$Q_l f(x) = Q_{l_1} Q_{l_2} \cdots Q_{l_n} f(x) = \left(\prod_{i=1}^n Q_{l_i} \right) f(x).$$

By Lemma 2.6 of [13], we have

$$\bigvee_i^{I^n} Q_l f = \bigvee_i^{I^n}\left(\prod_{j=1}^n Q_{l_j}\right) f = \bigvee_i^{I^n} Q_{l_i}\left(\prod_{j=1, j\neq i}^n Q_{l_j}\right) f \leqslant \bigvee_i^{I^n}\left(\prod_{j=1, j\neq i}^n Q_{l_j}\right) f.$$

We now show that

$$\int_{I^{n-1}} \bigvee_i^{I^n}\left(\prod_{j=1, j\neq i}^n Q_l\right) f\left(\prod_{j=1, j\neq i}^n dx_j\right) \leqslant \int_{I^{n-1}} \bigvee_i^{I^n} f\left(\prod_{j=1, j\neq i}^n dx_j\right). \qquad (1)$$

To prove (1), consider, for any $0 = x_i^0 < x_i^1 < \cdots < x_i^{r-1} < x_i^r = 1$,

$$\sum_{k=1}^r \left| \prod_{j=1, j\neq i}^n Q_{l_j} f(x_1, ..., x_i^{k-1}, ..., x_n) - \prod_{j=1, j\neq i}^n Q_{l_j} f(x_1, ..., x_i^k, ..., x_n)\right|$$

$$= \sum_{k=1}^r \left| \prod_{j=1, j\neq i}^n \sum_{r_j=0}^{l-1} \frac{1}{\prod_{j=1, j\neq i}^n m(J_{r_j})}\right.$$

$$\times \int_{\prod_{j=1, j\neq i}^n J_{r_j}} (f(x_1, .., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^k, ..., x_n))$$

$$\left.\left(\prod_{j=1, j\neq i}^n dx_j\right)\left(\prod_{j=1, j\neq i}^n \chi_{J_{r_j}}(x_j)\right)\right|$$

$$\leqslant \sum_{k=1}^r \left(\prod_{j=1, j\neq i}^n \sum_{r_j=0}^{l-1} \frac{1}{\prod_{j=1, j\neq 1}^n m(J_{r_j})}\right.$$

$$\times \int_{\prod_{j=1, j\neq i}^n J_{r_j}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^k, ..., x_n)|$$

$$\left.\times\left(\prod_{j=1, j\neq i}^n dx_j\right)\left(\prod_{j=1, j\neq i}^n \chi_{J_{r_j}}(x_j)\right)\right).$$

Now

$$\int_{I^{n-1}} \sum_{k=1}^r \left| \prod_{j=1, j\neq i}^n Q_{l_j} f(x_1, ..., x_i^{k-1}, ..., x_n)\right.$$

$$\left. - \prod_{j=1, j\neq i}^n Q_{l_j} f(x_1, ..., x_1^k, ..., x_n)\right|\left(\prod_{j=1, j\neq i}^n dx_j\right)$$

$$\leqslant \int_{I^{n-1}} \sum_{k=1}^{r} \left(\prod_{j=1, j\neq i}^{n} \sum_{r_j=0}^{l-1} \left(\prod_{j=1, j\neq i}^{n} m(J_{r_j}) \right)^{-1} \right.$$

$$\times \int_{\prod_{j=1, j\neq i}^{n} J_{r_j}} |f(\cdots x_i^{k-1} \cdots) - f(\cdots x_i^{k} \cdots)|$$

$$\left. \times \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \left(\prod_{j=1, j\neq i}^{n} \chi_{J_{r_j}}(x_j) \right) \right) \left(\prod_{j=1, j\neq i}^{n} dx_j \right)$$

$$= \sum_{k=1}^{r} \left(\prod_{j=1, j\neq i}^{n} \sum_{r_j=0}^{l-1} \int_{\prod_{j=1, j\neq i}^{n} J_{r_j}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \right.$$

$$\left. \times \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \frac{1}{\prod_{j=1, j\neq i}^{n} m(J_{r_j})} \int_{I^{n-1}} \prod_{j=1, j\neq i}^{n} \chi_{J_{r_j}}(x_j) \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \right)$$

$$= \sum_{k=1}^{r} \int_{I^{n-1}} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \left(\prod_{j=1, j\neq i}^{n} dx_j \right)$$

$$= \int_{I^{n-1}} \sum_{k=1}^{r} |f(x_1, ..., x_i^{k-1}, ..., x_n) - f(x_1, ..., x_i^{k}, ..., x_n)| \left(\prod_{j=1, j\neq i}^{n} dx_j \right).$$

Hence,

$$\int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{j=1, j\neq i}^{n} Q_{l_j} \right) f \left(\prod_{j=1, j\neq i}^{n} dx_j \right) \leqslant \int_{I^{n-1}} \bigvee_i^{I^n} f \left(\prod_{j=1, j\neq i}^{n} dx_j \right).$$

Now,

$$\mathbb{V}_i^{I^n} Q_l f = \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} h \left(\prod_{j=1, j\neq i}^{n} dx_j \right), h = Q_l f \text{ a.e.}, \bigvee_i^{I^n} h \text{ measurable} \right\}$$

$$\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} Q_l g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), g = f \text{ a.e.}, \bigvee_i^{I^n} Q_l g \text{ measurable} \right\}$$

$$\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} \left(\prod_{j=1, j\neq i}^{n} Q_{l_j} \right) g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), \right.$$

$$\left. g = f \text{ a.e.}, \bigvee_i^{I^n} g \text{ measurable} \right\}$$

$$\leqslant \inf \left\{ \int_{I^{n-1}} \bigvee_i^{I^n} g \left(\prod_{j=1, j\neq i}^{n} dx_j \right), g = f \text{ a.e.}, \bigvee_i^{I^n} g \text{ measurable} \right\} = \mathbb{V}_i^{I^n} f.$$

Therefore $\mathbb{V}^{I^n} Q_l f = \max_i \mathbb{V}_i^{I^n} Q_l f \leqslant \max_i \mathbb{V}_i^{I^n} f = \mathbb{V}^{I^n} f$. In the foregoing argument we have used the fact that for any positive integer l and $f, g \in L^1$, $f = g$ a.e. implies $Q_l f = Q_l g$ a.e. and $\bigvee_i^{I^n} g$ measurable implies $\bigvee_i^{I^n} Q_l g$ measurable. ▮

LEMMA 7. *Let τ be a Jablonski transformation*

$$\tau(x) = (\varphi_{1j}(x_j), ..., \varphi_{nj}(x_n)), \quad x \in D_j$$

and $f_l \in \Delta_l$ be any fixed point of $P_l(\tau)$ with $\| f_l \| = 1$. If

$$\lambda = \inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 2,$$

then the sequence $\{ \bigvee^{I^n} f_l \}_{l=1}^{\infty}$ is bounded.

Proof. By Lemma 3, $f_l = P_l f_l = Q_l P_\tau f_l$ for all l. Hence by Theorem 2 and Lemma 6.

$$\bigvee^{I^n} f_l = \bigvee^{I^n} Q_l P_\tau f_l \leqslant \bigvee^{I^n} P_\tau f_l \leqslant K_\tau \| f_l \| + \alpha \bigvee^{I^n} f_l = K_\tau + \alpha \bigvee^{I^n} f_l,$$

where $K_\tau > 0$ and $0 < \alpha < 1$. Since $\bigvee^{I^n} f_l < \infty$, we have $\bigvee^{I^n} f_l \leqslant K_\tau/(1-\alpha)$. ∎

The following self-adjoint property of Q_l which was not needed in [13] plays a vital role in the sequel.

LEMMA 8. *For any $f \in L^1$, $l = 1, 2, ...,$ and measurable subset A of I^n*

$$\int_{I^n} \chi_A Q_l f \, dx = \int_{I^n} f Q_l \chi_A \, dx.$$

Proof.

$$\begin{aligned}
\int_{I^n} \chi_A(x) Q_l f(x) \, dx &= \int_{I^n} \chi_A(x) \left(\sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} f(y) \, dy \, \chi_k(x) \right) dx \\
&= \sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} f(y) \, dy \int_{I^n} \chi_A(x) \chi_k(x) \, dx \\
&= \sum_{k=1}^{l^n} \frac{m(A \cap I_k)}{m(I_k)} \int_{I_k} f(y) \, dy = \sum_{k=1}^{l^n} \frac{m(A \cap I_k)}{m(I_k)} \int_{I_k} f(x) \, dx \\
&= \int_{I^n} f(x) \left(\sum_{k=1}^{l^n} \frac{1}{m(I_k)} \int_{I_k} \chi_A(y) \, dy \, \chi_k(x) \right) dx \\
&= \int_{I^n} f(x) Q_l \chi_A(x) \, dx. \quad ∎
\end{aligned}$$

THEOREM 3. *Let τ be a nonsingular Jablonski transformation with partition $\{D_1, ..., D_p\}$ and $\lambda = \inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 2$. Suppose P_τ has a unique fixed point. Then for any positive integer l, $P_l(\tau)$ has a fixed point f_l in Δ_l*

with $\|f_l\|=1$ *and the sequence* $\{f_l\}$ *converges weakly to the fixed point of* P_τ.

Proof. By Lemma 7 and Lemma 3 of [9], we know that the set $\{f_l\}_{l=1}^{\infty}$ is weakly relatively compact in L^1. Let $\{f_{l_j}\}$ be any weakly convergent subsequence of $\{f_l\}_{l=1}^{\infty}$ and let $f=\lim_{j\to\infty} f_{l_j}$ weakly. Then for any $g\in L^\infty$,

$$\left|\int_{I^n} g(f-P_\tau f)\,dx\right| \leqslant \left|\int_{I^n} g(f-f_{l_j})\,dx\right| + \left|\int_{I^n} g(f_{l_j}-Q_{l_j}P_\tau f_{l_j})\,dx\right| + \left|\int_{I^n} g(Q_{l_j}P_\tau f_{l_j}-P_\tau f)\,dx\right|.$$

The first term approaches 0 since f_{l_j} converges weakly to f as $j\to\infty$. By Lemma 3, $Q_{l_j}P_\tau f_{l_j}=P_{l_j}f_{l_j}=f_{l_j}$. The second term is identically 0.

We now consider the last term. By the weak continuity of P_τ [11, p. 43], $P_\tau f_{l_j}$ converges weakly to $P_\tau f$ as $j\to\infty$. We will prove that $Q_{l_j}P_\tau f_{l_j}$ converges weakly to $P_\tau f$ as $j\to\infty$. It is enough to show that for any measurable subset A of I^n we have

$$\lim_{j\to\infty}\int_{I^n}\chi_A Q_{l_j}h_{l_j}\,dx=\int_{I^n}\chi_A h\,dx,$$

where $h_{l_j}=P_\tau f_{l_j}$ and $h=P_\tau f$.

By Corollary IV.8.11 in [4, p. 294],

$$\int_E h_{l_j}(x)\,dx\to 0 \text{ as } m(E)\to 0$$

uniformly in j. Because $\|h_{l_j}\|=1$ and $h_{l_j}\geqslant 0$, by Theorem 7.5.3 in [1, p. 296], h_{l_j} are uniformly integrable, i.e.,

$$\int_{\{|h_{l_j}|\geqslant K\}}|h_{l_j}|\,dx\to 0 \qquad \text{as} \quad K\to\infty$$

uniformly in j. Therefore, for any $\varepsilon>0$, there exists $K>0$ such that for all j

$$2\int_{\{|h_{l_j}|\geqslant K\}}|h_{l_j}|\,dx<\varepsilon.$$

Hence

$$\left|\int_{I^n} h_{l_j}(Q_{l_j}\chi_A-\chi_A)\,dx\right|$$

$$\leqslant\int_{I^n}|h_{l_j}|\,|Q_{l_j}\chi_A-\chi_A|\,dx$$

$$= \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, |Q_{l_j}\chi_A - \chi_A| \, dx + \int_{\{|h_{l_j}| < K\}} |h_{l_j}| \, |Q_{l_j}\chi_A - \chi_A| \, dx$$

$$\leq 2 \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, dx + K \int_{\{|h_{l_j}| < K\}} |Q_{l_j}\chi_A - \chi_A| \, dx$$

$$\leq 2 \int_{\{|h_{l_j}| \geq K\}} |h_{l_j}| \, dx + K \int_{I^n} |Q_{l_j}\chi_A - \chi_A| \, dx.$$

The first term is less than ε and the second term approaches 0 as $j \to \infty$ by Lemma 2. Thus

$$\lim_{j \to \infty} \int_{I^n} h_{l_j}(Q_{l_j}\chi_A - \chi_A) \, dx = 0.$$

By Lemma 8,

$$\lim_{j \to \infty} \int_{I^n} \chi_A Q_{l_j} h_{l_j} \, dx = \lim_{j \to \infty} \int_{I^n} h_{l_j} Q_{l_j} \chi_A \, dx$$

$$= \lim_{j \to \infty} \int_{I^n} h_{l_j}(Q_{l_j}\chi_A - \chi_A) \, dx + \lim_{j \to \infty} \int_{I^n} h_{l_j} \chi_A \, dx$$

$$= \int_{I^n} h \chi_A \, dx.$$

This means the last term approaches 0.

We have, therefore, established that for any $g \in L^\infty$,

$$\int_{I^n} g(x) \, (f(x) - P_\tau f(x)) \, dx = 0.$$

This means $P_\tau f(x) = f(x)$ almost everywhere. Therefore any weakly convergent subsequence of $\{f_l\}$ converges weakly to a unique fixed point of P_τ. Hence $f_l \to f$ weakly as $l \to \infty$. ∎

COROLLARY 1. *If the fixed point of P_τ is not unique in Theorem 3, then any weak limit point of $\{f_l\}_{l=1}^\infty$ is a fixed point of P_τ.*

THEOREM 4. *Let τ be a nonsingular Jablonski transformation with $\lambda = \inf_{i,j} \{\inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}|\} > 1$. Suppose P_τ has a unique fixed point. Let k be an integer such that $\lambda^k > 2$. Let $\phi = \tau^k$ and f_l be a fixed point of $P_l(\phi)$. Let*

$$g_l = \frac{1}{k} \sum_{j=0}^{k-1} P_{\tau^j} f_l.$$

Then $\{g_l\}$ converges weakly to the fixed point of P_τ.

Proof. Since P_τ is a weakly continuous operator [11, p. 43], Theorem 3 implies that $g_l \to g = (1/k)\sum_{j=0}^{k-1} P_{\tau^j} f$ weakly as $l \to \infty$. Therefore

$$P_\tau g = \frac{1}{k}\sum_{j=1}^{k} P_{\tau^j} f = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f = g,$$

where f is the fixed point of $P_\phi = P_{\tau^k}$, i.e., $P_{\tau^k} f = f$. ∎

COROLLARY 2. *If the fixed point of P_τ is not unique in Theorem 4, then any weak limit point f of $\{f_l\}_{l=1}^{\infty}$ is a fixed point of P_ϕ and $g = (1/k)\sum_{j=0}^{k-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{l_i} \to f$ weakly as $i \to \infty$ then $g_{l_i} = (1/k)\sum_{j=0}^{k-1} P_{\tau^j} f_{l_i} \to g$ weakly as $i \to \infty$.*

4. UNIQUENESS OF INVARIANT DENSITIES

Let $\tau: I^n \to I^n$ be a Jablonski transformation. Without loss of generality we shall assume there exist

$$0 = a_{i,0} < a_{i,1} < \cdots < a_{i,r_i} = 1, \qquad i = 1, 2, ..., n$$

for some positive integers $r_1, r_2, ..., r_n$ such that the partition β is composed of sets $D_{s_1, ..., s_n} = \prod_{i=1}^n D_{s_i}$, where $D_{s_i} = [a_{i,s_i-1}, a_{i,s_i})$, $s_i = 1, 2, ..., r_i - 1$, $D_{r_i} = [a_{i,r_i-1}, a_{i,r_i}]$, and τ is given by the formula

$$\tau(x) = (\varphi_{1,s_1,...,s_n}(x_1), ..., \varphi_{n,s_1,...,s_n}(x_n)), x \in D_{s_1,...,s_n},$$

where $\varphi_{i,s_1,...,s_n}: \bar{D}_{s_i} \to [0, 1]$ are C^2 functions.

DEFINITION 3. We say that the partition β has the *communication property* under the transformation $\tau: I^n \to I^n$ if for any elements $D'_{s_1,...,s_n}$ and $D''_{s_1,...,s_n}$ of β there exist integers u and v such that $D'_{s_1,...,s_n} \subset \tau^u(D''_{s_1,...,s_n})$ and $D''_{s_1,...,s_n} \subset \tau^v(D'_{s_1,...,s_n})$.

DEFINITION 4. A Jablonski transformation $\tau: I^n \to I^n$ is in class $\mathscr{C}$ if it satisfies following conditions for the fixed partition β:

(1) $\inf|\varphi_i'| > 0$ and $\inf|(\varphi_i^w)'| > 1$ for some integer w;

(2) τ is piecewise C^2;

(3) the partition β has the communication property under τ.

We associate with each $D_{s_1,...,s_n}$ a symbol such as α, β, γ, ..., and code the orbit by a sequence $\langle x \rangle = \alpha\beta\gamma\cdots$ if $x \in D(\alpha)$, $\tau(x) \in D(\beta)$, $\tau^2(x) \in D(\gamma)$, ..., where $D(\alpha)$ is some $D_{s_1,...,s_n}$ whose symbol is α. The following three lemmas are identical to the one-dimensional versions proved in [2].

LEMMA 9. *Let $\tau: I^n \to I^n$ be a C^2 Jablonski transformation which satisfies condition* (1) *defining class* $\mathscr{C}$. *Then* $\langle x \rangle = \langle y \rangle$ *implies* $x = y$.

LEMMA 10. *Let τ be as in Lemma* 9. *If $\sigma = \alpha_1 \alpha_2 \cdots$ is a sequence with the property that $\tau(D(\alpha_k)) \supset D(\alpha_{k+1})$, $k = 1, 2, \ldots$, then there exists a unique $x \in I^n$ such that $\langle x \rangle = \sigma$.*

LEMMA 11. *Let τ be the same as in Lemma* 9 *and let $\xi \subset \beta$ be a collection of elements satisfying the communication property: for any $D_1, D_2 \in \xi$, there exist integers u and v such that $D_1 \subset \tau^u(D_2)$ and $D_2 \subset \tau^v(D_1)$. Assume that ξ contains at least two D_i's and $V = \bigcup_{D \in \xi} D$. Then there exists an $x \in V$ such that $\{\tau^l(x)\}_{l=1}^{\infty}$ is dense in V.*

LEMMA 12. *If τ is the same as in Lemma* 11 *and satisfies condition* (3) *defining class* $\mathscr{C}$, *then there exists a dense orbit in all of I^n.*

THEOREM 5. *If $\tau \in \mathscr{C}$ then the absolutely continuous invariant measure under τ is unique.*

Proof. Assume there exist two such measure with densities f_1 and f_2. As in [14], it can be shown that there exist two invariant functions $f_1^* \geqslant 0$, $f_2^* \geqslant 0$, $\| f_1^* \| = \| f_2^* \| = 1$ such that $S_1 = \operatorname{spt} f_1^*$ and $S_2 = \operatorname{spt} f_2^*$ are disjoint and S_i is an union of disjoint regions, $i = 1, 2$. From [8] we know that each S_i has interior.

Now let $x \in I^n$ be a point which has a dense orbit in I^n. By Lemma 12 such a point exists. The denseness of the orbit $\{\tau^l(x)\}_{l=1}^{\infty}$ implies there exist points $u = \tau^{l_1}(x)$ and $v = \tau^{l_2}(u)$ such that $u \in \operatorname{int} S_1$ and $v \in \operatorname{int} S_2$, where int denotes interior. By the piecewise continuity of τ there exists an open ball O_1 centered at u and in S_1 such that for $u \in O_1$, $v = \tau^{l_2}(u) \in \operatorname{int} S_2$. But S_1 and S_2 are invariant sets [14], i.e., $\tau(S_i)$ a.e., $i = 1, 2$. Hence, we have a contradiction. Therefore, there exists only one absolutely continuous invariant measure under τ. ∎

THEOREM 6. *Let $\tau: I^n \to I^n$ be a Jablonski transformation with the partition $\beta = \{D_{s_1, \ldots, s_n}\}$, $s_i = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, n$ given by the formula*

$$\tau(x) = (\varphi_{1, s_1, \ldots, s_n}(x_1), \ldots, \varphi_{n, s_1, \ldots, s_n}(x_n)),\ x \in D_{s_1, \ldots, s_n}$$

such that: (1) *for any $s_1, \ldots, s_n$ and i, $\varphi_{i, s_1, \ldots, s_n}(x_i) \in C^2$ and $|\varphi'_{i, s_1, \ldots, s_n}(x_i)| \geqslant \lambda > 1$; and* (2) *the partition β has the communication property with respect to τ. Then P_τ has a unique fixed point f with $\| f \| = 1$.*

Proof. *τ satisfies all the conditions of Theorem* 5. ∎

5. Examples

(1) If for any element $D_{s_1, \ldots, s_n}$ of β, $\varphi_{i, s_1, \ldots, s_n}$ is a C^2 bijective map of the closed interval $\bar{D}_{s_i}$ onto $[0, 1]$, the restriction $\tau_{s_1, \ldots, s_n}$ of τ on $D_{s_1, \ldots, s_n}$ is a C^2 bijective transformation of $\bar{D}_{s_1, \ldots, s_n}$ onto I^n and

$$\lambda = \inf | \varphi'_{i, s_1, \ldots, s_n} | > 1,$$

then $\tau \in \mathscr{C}$ and by Theorem 5 the absolutely continuous invariant measure under τ is unique. Furthermore, If $\lambda > 2$ then by Theorem 3 we have a sequence of piecewise constant functions f_l with $\| f_l \| = 1$ which converges weakly to the fixed point of P_τ.

(2) We now present an example for $n = 2$, where the elements of the partition β do not map onto all of I^2. (See Fig. 1.) Let $I_1 = J_1 = [0, \frac{1}{4})$, $I_2 = J_2 = [\frac{1}{4}, \frac{1}{2})$, $I_3 = J_3 = [\frac{1}{2}, \frac{3}{4})$, $I_4 = J_4 = [\frac{3}{4}, 1]$ and $D_{kj} = I_k \times J_j$, k, $j = 1, 2, 3, 4$.
Let $h_1(x) = 2.4(x^2 + x)$, $h_2(x) = h_1(x - \frac{1}{4})$, $h_3(x) = h_1(x - \frac{1}{2})$, $h_4(x) = h_1(x - \frac{3}{4})$, $g_i(y) = h_i(y)$, $i = 1, 2, 3, 4$, $h(x) = 4x$, $g(y) = 4y$. Then define

$$\tau(x, y) = \begin{cases} (h_k(x), g_j(y)), & (x, y) \in D_{kj}, \quad D_{kj} \neq D_{11}, \\ (h(x), g(y)), & (x, y) \in D_{11}. \end{cases}$$

Since $\tau(\bar{D}_{kj}) = [0, \frac{3}{4}] \times [0, \frac{3}{4}]$, $(D_{kj} \neq D_{11})$, $\tau(\bar{D}_{11}) = I^2$. By [8] we know that P_τ has a fixed point and by Theorem 6 it is unique. Also in view of Theorem 3, $f_l \in \Lambda_l$ with $\| f_l \| = 1$ and $\{f_l\}$ converges weakly to the fixed point of P_τ as $l \to \infty$.

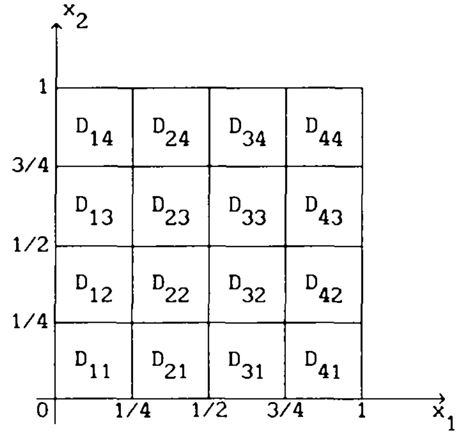

FIG. 1. The domain of a two-dimensional Jablonski transformation.

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